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G-ORDERED FUNCTIONS, WITH APPLICATIONS IN STATISTICS, II. APPLICATIONS.

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ABSTRACT

This is Part II of a two-part paper which continues the unification of stochastic comparisons. Many commonly used multivariate densities are shown to be G-ordered and, in fact, each such density may be used as the kernel function in the integral transform for the preservation of G-monotonicity. We show that any elliptically-contoured density is G-ordered. We present an application of G-ordered functions to certain well-known tests of a multivariate hypothesis. Sufficient conditions on the distribution of the observations are determined so that the tests have G-monotone increasing power functions.

1. Introduction and Summary.

This is Part II of a two-part paper which generalizes a rearrangement ordering, develops the theory of functions isotonic with respect to the more general ordering, and presents applications in statistics. In Part I we defined reflection ordering (a generalization of transposition ordering) and G-ordered functions (a generalization of functions decreasing in transposition (DT)). (See Hollander, Proschan, and Sethuraman (1977) for definitions of transposition ordering and DT functions.) In Part II we present applications of G-ordered functions in statistics.

In Section 2 we show that many well-known multivariate densities have the G-ordered property for G, the permutation group. In fact, each density may be used as the kernel K for the preservation of G-monotonicity under the integral transform

$$\int K(\lambda,x)\, f(x)\, d\mu(x).$$

We show that the class of densities proportional to $\exp\left\{-\frac{1}{\alpha}\sum_{i=1}^{n}|x_i-\lambda_i|^{\theta}\right\}$, where $\alpha,\theta > 0$, has the G-ordered property for G, the group of sign changes and permutations. Finally we determine the reflection group G for which any elliptically-contoured density has the G-ordered property.

In Section 3 we apply the theory of G-ordered functions to a class of hypothesis testing problems. For a given testing problem we determine sufficient conditions on the model such that the power functions of a well-defined class of tests have a G-monotone or a G-ordered property. We apply our results to both parametric and non-parametric models, each of which contains a wide variety of linear models as well.

2. G-ordered Densities in Statistics.

In the first part of this section we show that each of a number of well-known densities may be used as the kernel function of the integral transform for the preservation of G-monotonicity. We make use of the main theorem for the preservation of G-monotone functions and its corollaries (Theorem 4.15, Corollaries 4.17, 4.18, 4.19, 4.20, and 4.21 of Part I). In the second part of this section we discuss elliptically-contoured functions. These functions possess the G-ordered property for some reflection group. Notable examples of elliptically-contoured densities are the multivariate normal, the multivariate T, and the multivariate Cauchy.

Throughout this section and the section to follow, let $\mathcal{K}_{G,\mu}$ be the class of all kernels K on R^{2n} which preserve G-monotonicity under the following integral transform with respect to the measure μ:

$$h(\lambda) = \int K(\lambda,x)\, f(x)\, d\mu(x).$$

Let R^{n+} and Z^{n+} denote the set of points in Euclidean n-space whose coordinates are nonnegative real numbers and nonnegative integers respectively.

Theorem 2.1. Let K_i, defined on $\Lambda_i \times \mathcal{X}_i$, $i = 1,2,\ldots,7$, be the i^{th} density listed in 1-7 below. Let G be the permutation group acting on R^n and let μ be the counting measure on $\mathcal{X}_i$. Then $K_i \in \mathcal{K}_{G,\mu}$ for $i = 1,2,\ldots,7$.

(1). Multinomial.

$K_1(\lambda,x) = N! \prod_{i=1}^{n} \frac{\lambda_i^{x_i}}{x_i!}$. The set $\Lambda_1 = R^{n+}$ and the set

$\mathcal{X}_1 = \{x \in R^n : x_i = 0,1,\ldots,n,\ \ i = 1,2,\ldots,n,\ \text{and}\ \sum_{i=1}^{n} x_i = N\}$.

(2). <u>Negative Nultinomial.</u>

$$K_2(\lambda,x) = \frac{\Gamma(N + \sum_{i=1}^{n} x_i)}{\Gamma(N)} \left(1 + \sum_{i=1}^{n} \lambda_i\right)^{-N-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \frac{\lambda_i^{x_i}}{x_i!}.$$ The set $\Lambda_2 = R^{n+}$ and the set $X_2 = Z^{n+}$.

(3). <u>Multivariate Poisson.</u>

$$K_3(\lambda,x) = e^{-(\xi + \lambda_1 + \ldots + \lambda_n)} \sum_{j=1}^{\min_{1\le i\le n} x_i} \frac{\xi^j}{j!} \prod_{i=1}^{n} \frac{\lambda_i^{x_i - j}}{(x_i - j)!}.$$ The set $\Lambda_3 = R^{n+}$ and the set $X_3 = Z^{n+}$.

(4). <u>Multivariate Hypergeometric.</u>

$$K_4(\lambda,x) = \prod_{i=1}^{n} \binom{\lambda_i}{x_i} \Big/ \binom{\sum_{i=1}^{n} \lambda_i}{N}.$$ The set $\Lambda_4 = R^{n+}$ and the set $X_4 = X_\lambda =$

$\{x \in R^n : x_i = 0,1,\ldots,N,\ i = 1,2,\ldots,n,$ and $\sum_{i=1}^{n} x_i = N < \sum_{i=1}^{n} \lambda_i\}$.

(5). <u>Negative Multivariate Hypergeometric.</u>

$$K_5(\lambda,x) = \frac{N!\,\Gamma\left(\sum_{i=1}^{n} \lambda_i\right)}{\prod_{i=1}^{n} x_i!\ \Gamma\left(N + \sum_{i=1}^{n} \lambda_i\right)} \prod_{i=1}^{n} \frac{\Gamma(\lambda_i + x_i)}{\Gamma(\lambda_i)}.$$ The set $\Lambda_5 = R^{n+}$ and the set

$X_5 = \{x \in R^n : x_i = 0,1,\ldots,N,\ i = 1,2,\ldots,n,$ and $\sum_{i=1}^{n} x_i = N\}$.

(6). <u>Dirichlet Compound Negative Multinomial.</u>

$$K_6(\lambda,x) = \frac{\Gamma\left(N + \sum_{i=1}^{n} x_i\right)\Gamma\left(\theta + \sum_{i=1}^{n} \lambda_i\right)\Gamma(N+\theta)}{\prod_{i=1}^{n} x_i!\ \Gamma(N)\,\Gamma(\theta)\,\Gamma\left(N + \theta + \sum_{i=1}^{n} (\lambda_i + x_i)\right)} \prod_{i=1}^{n} \frac{\Gamma(\lambda_i + x_i)}{\Gamma(\lambda_i)}.$$ The set $\Lambda_6 = R^{n+}$

and the set $X_6 = Z^{n+}$.

(7). Dirichlet Compound Multinomial.

$$K_7(\lambda,x) = \frac{N!\,\Gamma\left(\sum_{i=1}^{n} \lambda_i\right)}{\Gamma\left(N + \sum_{i=1}^{n} \lambda_i\right)} \prod_{i=1}^{n} \frac{\Gamma(\lambda_i + x_i)}{\Gamma(\lambda_i)}.$$ The set $\Lambda_7 = R^{n+}$ and the set

$$X_7 = \{x \in R^n : x_i = 0,1,\ldots,n,\ i = 1,2,\ldots,n, \text{ and } \sum_{i=1}^{n} x_i = N\}.$$

Proof. Note that μ, the counting measure, is translation invariant and is also G-invariant when G is the permutation group acting on R^n. For $i = 1,2,\ldots,7$, Λ_i and X_i are G-invariant subsets of R^n.

(1). Let $\phi(\lambda,x)$ be the density of n independent Poisson random variables with parameters $\lambda_1,\lambda_2,\ldots,\lambda_n$, and note that ϕ has the generalized semigroup property with respect to counting measure. Define $\ell(x) = \sum_{i=1}^{n} x_i$ and

$K(\ell(\lambda), \ell(x) = \frac{1}{\tilde{K}(\ell(\lambda), \ell(x))}$, where $\tilde{K}$ is the density of a univariate Poisson random variable with parameter $\ell(\lambda)$. Define the transformation $T\lambda = \left(\frac{\lambda_1}{\ell(\lambda)}, \frac{\lambda_2}{\ell(\lambda)}, \ldots, \frac{\lambda_n}{\ell(\lambda)}\right)$ and the function $K_1(T\lambda,Tx) = \phi(\lambda,x)\, K(\ell(\lambda), \ell(x))$, so that K_1 is the density of a multinomial random variable with parameter $T\lambda$. Clearly $\lambda \overset{G}{\geq} \tilde{\lambda}$ if and only if $T\lambda \overset{G}{\geq} T\tilde{\lambda}$. We thus conclude that $K_1 \in \mathcal{K}_{G,\mu}$ by Corollary 4.19 of Part I.

(2) We obtain the negative multinomial by mixing n independent Poisson random variables according to a gamma distribution. Thus K_2 has the G-ordered conditional generalized semigroup property and consequently $K_2 \in \mathcal{K}_{G,\mu}$ by Corollary 4.17 of Part I.

(3). Suppose that $U, X_1, X_2, \ldots, X_n$ are independent Poisson random variables with parameters $\xi, \lambda_1, \lambda_2, \ldots, \lambda_n$ respectively. Define $Y_i = U + X_i$, $i = 1,2,\ldots,n$. Then K_3 is the joint density of $Y_1, Y_2, \ldots, Y_n$. The conditional joint density $K_u(\lambda, x)$ of $Y_1, Y_2, \ldots, Y_n$ has the G-ordered generalized semigroup property. Thus K_3 has the G-ordered conditional generalized semigroup property and consequently $K_3 \in \mathcal{K}_{G,\mu}$ by Corollary 4.17 of Part I.

(4). The multivariate hypergeometric distribution is the conditional distribution of n independent binomial random variables given their sum. An argument analogous to that which we used to show that $K_1 \in \mathcal{K}_{G,\mu}$ may be used to show that $K_4 \in \mathcal{K}_{G,\mu}$.

(5). Let $\phi(\beta, x)$ be the multinomial density and $\psi(\beta, \lambda)$ be the Dirichlet density. Then $K_5(\lambda, x) = \int \psi(\beta, \lambda)\, \phi(\beta, x)\, d\mu^*(x)$, where μ^* is Lebesgue measure. We establish in Theorem 2.2 below that $\psi \in \mathcal{K}_{G,\mu^*}$. Thus as a consequence of Corollary 4.20 of Part I we conclude that $K_5 \in \mathcal{K}_{G,\mu}$.

(6) and (7). We invoke Corollary 4.20 to conclude that $K_6, K_7 \in \mathcal{K}_{G,\mu}$.||

<u>Theorem 2.2</u>. Let K_i, defined on $\Lambda_i \times \mathcal{X}_i$, $i = 1,2,\ldots,5$, be the i^{th} density listed in 1-5 below. Let G be the permutation group acting on R^n and let μ be Lebesgue measure on $\mathcal{X}_i$. Then $K_i \in \mathcal{K}_{G,\mu}$ for $i = 1,2,\ldots,5$.

(1). <u>Dirichlet</u>.

$$K_1(\lambda, x) = \frac{\Gamma\left(\theta + \sum_{i=1}^{n} \lambda_i\right)}{\Gamma(\theta) \prod_{i=1}^{n} \Gamma(\lambda_i)} \left(1 - \sum_{i=1}^{n} x_i\right)^{\theta-1} \prod_{i=1}^{n} x_i^{\lambda_i - 1}.$$

The set $\Lambda_1 = R^{n+}$ and the set

$\mathcal{X}_1 = \{x \in R^n : x_i \geq 0,\ i = 1,2,\ldots,n, \text{ and } \sum_{i=1}^{n} x_i \leq 1\}$.

(2). **Inverted Dirichlet.**

$$K_2(\lambda,x) = \frac{\Gamma\left(\theta + \sum_{i=1}^{n} \lambda_i\right) \prod_{i=1}^{n} x_i^{\lambda_i - 1}}{\Gamma(\theta) \prod_{i=1}^{n} \Gamma(\lambda_i) \left(1 + \sum_{i=1}^{n} x_i\right)^{\theta + \sum_{i=1}^{n} \lambda_i}}.$$ The set $\Lambda_2 = X_2 = R^{n+}$.

(3). **Multivariate Gamma.**

$$K_3(\lambda,x) = \int_0^{\min_{1\le i\le n} x_i} \frac{u^{\theta-1} \prod_{i=1}^{n} (x_i - u)^{\lambda_i - 1} \exp\{(n-1)u - \sum_{i=1}^{n} x_i\}}{\prod_{i=1}^{n} \Gamma(\lambda_i)} du.$$ The set

$\Lambda_3 = X_3 = R^{n+}$.

(4). **Multivariate F.**

$$K_4(\lambda,x) = \frac{\Gamma(\lambda_o) \prod_{i=0}^{n} (2\lambda_i)^{\lambda_i} \prod_{i=1}^{n} x_i^{\lambda_i - 1}}{2 \prod_{i=0}^{n} \Gamma(\lambda_i) \left(\lambda_o + \sum_{i=1}^{n} \lambda_i x_i\right)^{\sum_{i=0}^{n} \lambda_i}}.$$ The set $\Lambda_4 = X_4 = R^{n+}$.

(5). **Multivariate Normal.**

$$K_5(\lambda,x) = \frac{1}{(2\Pi)^{n/2} \left(\prod_{i=1}^{n} \lambda_i\right)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2} \sum_{i=1}^{n} \frac{x_i^2}{\lambda_i}\right\}$$ The set $\Lambda_5 = R^{n+}$ and the set $X_5 = R^n$.

Proof. Note that μ, the Lebesgue measure, is translation invariant and is also G-invariant when G is the permutation group acting on R^n. For $i = 1,2,\ldots,5$, Λ_i and X_i are G-invariant subsets of R^n.

Suppose $X_o, X_1, \ldots, X_n$ are independent chi-square random variables with $\lambda_o, \lambda_1, \ldots, \lambda_n$ degrees of freedom respectively. Let $Y_j = X_j\left(\sum_{i=0}^{n} X_i\right)^{-1}$, $j = 1,2,\ldots,n$. The joint density of $Y_1, Y_2, \ldots, Y_n$ is a Dirichlet density. We appeal to Corollary 4.21 of Part I to conclude that $K_1 \in \mathcal{K}_{G,\mu}$.

(2). Suppose $X_o, X_1, \ldots, X_n$ are as in (1) above. The joint density of $Y_j = X_j/X_o$, $j = 1,2,\ldots,n$, is an inverted Dirichlet distribution. The conditional distribution $K_u(\lambda, x)$ of $Y_1, Y_2, \ldots, Y_n$ given $X_o = u$ has the G-ordered generalized semigroup property with respect to Lebesgue measure. Consequently, K_2 has the G-ordered conditional generalized semigroup property and thus by Corollary 4.17 of Part I, $K_2 \in \mathcal{K}_{G,\mu}$.

(3) Suppose $X_o, X_1, \ldots, X_n$ are independent gamma random variables with respective scale parameters $\lambda_o, \lambda_1, \ldots, \lambda_n$ and common shape parameter θ. The joint density of $Y_j = X_o + X_j$, $j = 1,2,\ldots,n$ is multivariate gamma and it has the G-ordered conditional generalized semigroup property, so that $K_3 \in \mathcal{K}_{G,\mu}$ as a consequence of Corollary 4.17 of Part I.

(4). Suppose $X_o, X_1, \ldots, X_n$ are independent chi-square random variables with respective degrees of freedom $2\lambda_o, 2\lambda_1, \ldots, 2\lambda_n$. Let $Y_j = \dfrac{X_j/2\lambda_j}{X_o/2\lambda_o}$, $j = 1,2,\ldots,n$. The joint density of $Y_1, Y_2, \ldots, Y_n$ is the multivariate F density and it has the G-ordered conditional generalized semigroup property. Thus we use Corollary 4.17 of Part I to conclude that $K_4 \in \mathcal{K}_{G,\mu}$.

(5). The density K_5 has the G-ordered generalized semigroup property so that by Theorem 4.15 of Part I we conclude that $K_5 \in \mathcal{K}_{G,\mu}$.||

We note here that Hollander, Proschan, and Sethuraman (1977) have shown that the multivariate logarithmic series and the multivariate Pareto densities are G-ordered for the permutation group. We have been unable to determine if either of these densities are elements of $\mathcal{K}_{G,\mu}$ for any translation invariant and G-invariant measure μ.

Consider the class of densities of the form:

$$K(\lambda,x) = c \cdot \exp\left\{-\frac{1}{\alpha}\sum_{i=1}^{n}|x_i - \lambda_i|^{\theta}\right\},\ \alpha,\theta > 0.$$

When $\theta = 1$ and $c = (2\alpha)^{-n}$, $K(\lambda,x)$ is the joint density of n independent random variables from the univariate Laplace or double exponential distribution. Densities of this form are G-ordered for the group of permutations and sign changes. Stated formally:

__Theorem 2.3__. Let $K(\lambda,x) = c \cdot \exp\left\{-\frac{1}{\alpha}\sum_{i=1}^{n}|x_i - \lambda_i|^{\theta}\right\}$, where $\alpha,\theta > 0$ and $x_i,\lambda_i \in R^1$, $i = 1,2,\ldots,n$. Let G be the group of permutations and sign changes. Then K is G-ordered.

__Proof__. Since K is of the form $f(x - \lambda)$, it is equivalent to show that $f(x) = c \cdot \exp\left\{-\frac{1}{\alpha}\sum_{i=1}^{n}|x_i|^{\theta}\right\}$ is G-monotone decreasing. Since $\exp\left\{-\frac{1}{\alpha}\sum_{i=1}^{n}|x_i|^{\theta}\right\}$ is a decreasing function of $\sum_{i=1}^{n}|x_i|^{\theta}$, it suffices to show that $g(x) = \sum_{i=1}^{n}|x_i|^{\theta}$ is G-monotone increasing. Now g is a smooth G-invariant function, so that g is

G-monotone increasing if and only if $(r'x)(r'\nabla g) \geq 0$ for all $x \in R^n$ and all $r \in \Delta_G^*$. A fundamental set of roots for G, Δ_G^*, is the set $\{r_i,\ i = 1,2,\ldots,n\} \cup \{r_{i,i+1},\ i = 1,2,\ldots,n-1\}$, where $r_i' = (0,\ldots,0,1,0,\ldots,0)$ and $r_{i,i+1}' = (0,\ldots,0,1/\sqrt{2},-1/\sqrt{2},0,\ldots,0)$. It is easy to check that $(r'x)(r'\nabla g) \geq 0$ for all $x \in R^n$ and all $r \in \Delta_G^*$. Thus f is G-monotone decreasing and as a consequence of Theorem 4.10, K is G-ordered.||

We now turn our attention to elliptically-contoured densities, which have the form $c \cdot f(x'Bx)$, where B is positive definite. For any quadratic form, $Q_B(x) \overset{\text{def}}{=} x'Bx$, with B positive definite, there exists a reflection group G_B for which Q_B is G-monotone increasing. If f is a decreasing function and $K(\lambda,x) = c \cdot f[(x-\lambda)'B(x-\lambda)]$, then K is G_B-ordered. We now present this result formally.

Lemma 2.5. Let B be an $N \times N$ matrix. Let $P = [r_1^{(1)} \ldots r_{n_1}^{(1)} \vdots \ldots \vdots r_1^{(k)} \ldots r_{n_k}^{(k)}] = [P_1 \vdots \ldots \vdots P_k]$ be a diagonalizer of B such that $\{r_1^{(i)},\ldots,r_{n_i}^{(i)}\}$ is a set of orthogonal eigenvectors corresponding to the eigenvalue λ_i and a basis for the subspace V_i, $i = 1,2,\ldots,k$. Let

$$D = \begin{bmatrix} O_1 & & \\ & O_2 & 0 \\ & 0 & \ddots & \\ & & & O_k \end{bmatrix},$$

where O_i is any $n_i \times n_i$ orthogonal matrix. Let $M = PDP'$. Then $MB = BM$.

<u>Proof</u>. Define

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & 0 & \\ & & \lambda_1 & & & & & \\ & & & \lambda_2 & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_2 & & \\ & 0 & & & & & \lambda_k & \\ & & & & & & & \ddots \\ & & & & & & & & \lambda_k \end{bmatrix},$$

and note that $D\Lambda = \Lambda D$. Now $MB = PDP'P\Lambda P' = PD\Lambda P' = P\Lambda DP' = P\Lambda P'PDP' = BM$, as desired.||

We now present a theorem which describes the group G for which the quadratic form, $Q_B(x) = x'Bx$, is G-monotone increasing. Denote the group G_B.

<u>Theorem 2.6</u>. Let V_i, $i = 1,2,\ldots,k$, be the subspaces of R^n whose bases are the sets of orthogonal left eigenvectors corresponding to the distinct eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_k$ of a positive definite matrix B. Then $G_B = O(V_1) \times O(V_2) \times \ldots \times O(V_k)$, where $O(V_i)$ is the orthogonal group acting on V_i, $i = 1,2,\ldots,k$.

<u>Proof</u>. Let the dimension of V_i be n_i and note that $O(V_i) = P_i O(R^{n_i}) P_i'$, $i = 1,2,\ldots,k$.

(i). We show that for any $g \in G_B$, $(gx)'B(gx) = x'Bx$. Now g is of the form PDP' as in Lemma 2.5, so that $gB = Bg$. Thus $(gx)'B(gx) = x'g'Bgx = x'g'gBx = x'Bx$.

(ii). We show that for $y \in C_{G_B}(x)$ (the convex hull of the G_B-orbit of x), $x'Bx \geq y'By$. Write $x = x_1 + x_2 + \ldots + x_k$ and $y = y_1 + y_2 + \ldots + y_k$, where $x_i, y_i \in V_i$, $i = 1,2,\ldots,k$. That $y \in C_{G_B}(x)$ is equivalent to $x_i'x_i \geq y_i'y_i$,

$i = 1,2,\ldots,k$. Now $x'BX = x'P\Lambda P'x = \sum_{i=1}^{k} x_i'P_i\Lambda_iP_i'x_i = \sum_{i=1}^{k} \lambda_i x_i'x_i \geq \sum_{i=1}^{k} \lambda_i y_i'y_i$

$= y'By$, where $\Lambda_i = \begin{bmatrix} \lambda_i & & 0 \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}$. Thus we have shown that $x'Bx \geq y'By$ whenever

$y \in C_{G_B}(x)$.||

<u>Application 2.7</u>. A general class of multivariate distributions has density function $K(\lambda,x)$ proportional to $\exp\{-v^{-1}[(x-\lambda)'B(x-\lambda)]^{v/2}\}$, with B positive definite and $v > 1$. (See Johnson and Kotz (1972) p. 298.) For a fixed B, any density of this form is G_B-ordered. Note that for $v = 2$, $K(\lambda,x)$ is the <u>multivariate normal density</u> with mean λ and variance-covariance matrix B^{-1}.

<u>Application 2.8</u>. Let X be a multivariate normal random vector with mean λ and variance-covariance matrix B^{-1}. Define $Y = (\sqrt{\nu}/S)X$, where S is a chi-square random variable with ν degrees of freedom, independent of X. The random vector Y with <u>multivariate T distribution</u> is G_B-ordered, as in Application 2.7. Note that when B is the identity matrix and the number of degrees of freedom ν is one, we have the <u>multivariate Cauchy density</u>. In this case $G_B = O(R^N)$.

3. Applications to Hypothesis Testing.

In this section we present some applications in hypothesis testing. The main thrust of the section is the demonstration of monotone properties of power functions.

Let X be an n-dimensional random vector with density $K(\lambda,x)$ on $\Lambda \times \mathcal{X}$, where Λ and $\mathcal{X}$ are subsets of V, a linear subspace of R^n. We wish to test either of the following hypotheses.

(1). $H_o : A_1'\lambda = 0$ versus $H_1 : A_1'\lambda \neq 0$.

(2). $H_o : A_1'\lambda \in \overline{F}$ versus $H_1 : A_1'\lambda \notin \overline{F}$.

Here $\overline{F}$ is a closed fundamental region for a group G and A_1 is an $n \times k$ matrix, $k \leq n$. The linear transformation on V defined by A_1' yields a space, $A_1'V$, whose dimension is at most the dimension of V. One obvious method for testing both (1) and (2) is to transform the space of observations by means of A' and base a test on some function of $A_1'x$, where x is an observation from $K(\lambda,x)$. Denote the density of $A_1'X$ by $\phi(A_1'\lambda, A_1'x)$ and suppose that it is an element of $K_{G,\mu}$ for some translation and G-invariant measure μ. Then any test of (1) based on $C_f = \{u \in A'\mathcal{X} : f(u) > c_\alpha\}$, where f is G-monotone increasing, has a G-monotone increasing power function. This is an immediate consequence of Theorem 4.15 and its corollaries in Part I. If the density ϕ is G-ordered, then any test of (2) based on $C_f = \{u \in A'\mathcal{X} : f(u) > c_\alpha\}$, where f is G-ordered with respect to $\overline{F}$, has a power function G-ordered with respect to $\overline{F}$. This is an immediate consequence of Corollary 3.15 of Part I.

Our main interest in this section lies in special cases of the hypothesis testing problems (1) and (2) above. Suppose that $A = [A_1 \vdots A_2]$ is an element of the orthogonal group acting on V and, for some reflection group G, $A'\Delta_G^* = [A_1 \vdots 0]' \Delta_G^* \cup [0 \vdots A_2]'\Delta_G^*$. Under certain assumptions on $K(\lambda,x)$, the density of X, the power functions of tests presented below have monotonic properties. We do not require that the distribution of the test statistic be computed nor do we need to determine if that distribution has a G-ordered property.

We now present some technical lemmas to set the stage for the main theorem embodying the results described above. We also summarize the discussion on the decomposition of reflection groups in Benson and Grove (1971). Many of the results below depend upon the decomposition of a reflection group.

As a preliminary, note that for any reflection group G acting on V, the group $\tilde{G} \overset{\text{def}}{=} \{A'gA : g \in G\}$, where $A \in O(V)$, is a reflection group acting on $A'V$.

Lemma 3.1. Let V be a subspace of R^n and suppose that $A \in O(V)$. Then for any reflection group G, $x \overset{G}{\geq} y$ if and only if $A'x \overset{\tilde{G}}{\geq} A'y$.

Lemma 3.2. Let V be a subspace of R^n and suppose that $A \in O(V)$. Then for any reflection group G, $g_1 \overset{\overline{F}_G}{\geq} g_2$ if and only if $A'g_1A \overset{\overline{F}_{\tilde{G}}}{\geq} A'g_2A$, where $\overline{F}_{\tilde{G}} = A'\overline{F}_G$.

Now we proceed with a short discussion on the decomposition of a reflection group. Let G be a reflection group acting on V. Suppose that Δ^* is a set of fundamental roots for G and that $\Delta^* = \Delta_1^* \cup \Delta_2^*$, with Δ_1^* and Δ_2^* nonempty and $\Delta_1^* \perp \Delta_2^*$. Let V_i be the subspace of V spanned by Δ_i^*, $i = 1,2$. Then the restriction $M_{r_1}|V_2$ is the identity transformation on V_2 for each $r_1 \in \Delta_1^*$; also, the restriction $M_{r_1}|V_1$ is the identity transformation on V_1 for each $r_1 \in \Delta_2^*$. Set $G_1 = \{g|V_1 : g \in G\}$ and $G_2 = \{g|V_2 : g \in G\}$, so that for $i = 1,2$, G_i is generated by the reflections

$M_{r_j} | V_1$ along the roots $r_j \in \Delta_1^*$. Each $g \in G$ can thus be expressed as $g|V_1 \oplus g|V_2$ acting on $V_1 \oplus V_2$. It follows then that G is isomorphic with $G_1 \times G_2$ acting on $V_1 \oplus V_2$.

We now present some additional lemmas whose proofs rely on the decomposition result above.

Lemma 3.3. Let G be a reflection group acting on V and let $A = [A_1 \vdots A_2]$ be an element of the orthogonal group acting on V such that $A'\Delta_G^* = [A_1 \vdots 0]'\Delta_G^* \cup [0 \vdots A_2]'\Delta_G^*$. Define $\tilde{G}_1 = \{[A_1 \vdots 0]'g[A_1 \vdots 0] : g \in G\}$ and $\tilde{G}_2 = \{[0 \vdots A_2]'g[0 \vdots A_2] : g \in G\}$. Then $\tilde{G}$, the reflection group generated by $A'\Delta_G^*$ is isomorphic with $\tilde{G}_1 \times \tilde{G}_2$ acting on $A'V = [A_1 \vdots 0]'V \oplus [0 \vdots A_2]'V$.

Proof. We have that $A'\Delta_G^* = [A_1 \vdots 0]'\Delta_G^* \cup [0 \vdots A_2]'\Delta_G^*$ and $[A_1 \vdots 0]'\Delta_G^* \perp [0 \vdots A_2]'\Delta_G^*$. Since for any $g \in G$, $[A_1 \vdots 0]'g[A_1 \vdots 0]$ is the restriction of $\tilde{g} \overset{def}{=} A'gA$ to the space $[A_1 \vdots 0]'V$ and $[0 \vdots A_2]'g[0 \vdots A_2]$ is the restriction of $\tilde{g}$ to the space $[0 \vdots A_2]'V$, we conclude that $\tilde{G}$ is isomorphic with $\tilde{G}_1 \times \tilde{G}_2$ acting on $[A_1 \vdots 0]'V \oplus [0 \vdots A_2]'V$.||

Lemma 3.4. Let G, $\tilde{G}$, $\tilde{G}_1$, $\tilde{G}_2$, and A be as in Lemma 3.3. Define $x^{(i)} = x_1^{(i)} + x_2^{(i)}$, $i = 1,2$, where $x_1^{(i)} \in [A_1 \vdots 0]'V$ and $x_2^{(i)} \in [0 \vdots A_2]'V$. Then the following statements are equivalent.

(1). $x_j^{(1)} \overset{\tilde{G}_j}{\geq} x_j^{(2)}$, $j = 1,2$.

(2). $x^{(1)} \overset{\tilde{G}}{\geq} x^{(2)}$.

(3). $Ax^{(1)} \overset{G}{\geq} Ax^{(2)}$.

Proof. That (2) and (3) are equivalent is a consequence of Lemma 3.1. To show the equivalence of (1) and (2), we note that the restriction $M_r|[A_1 \vdots 0]'V$ is the identity transformation on $[A_1 \vdots 0]'V$ for each $r \in [0 \vdots A_2]'\Delta_G^*$. Also the restriction $M_r|[0 \vdots A_2]'V$ is the identity transformation on $[0 \vdots A_2]'V$ for each $r \in [A_1 \vdots 0]'\Delta_G^*$. Thus $x_1^{(1)}$ and $x_1^{(2)}$ are equivalent under $\tilde{G}_2$-majorization and $x_2^{(1)}$ and $x_2^{(2)}$ are equivalent under $\tilde{G}_1$-majorization. Consequently, $x_j^{(1)} \overset{\tilde{G}_j}{\geq} x_j^{(2)}$, $j = 1,2$, if and only if $x^{(1)} \overset{\tilde{G}}{\geq} x^{(2)}$.||

Lemma 3.5. Let G, $\tilde{G}$, $\tilde{G}_1$, $\tilde{G}_2$, and A be as in Lemma 3.3. Define $\tilde{g}^{(i)} = \tilde{g}_1^{(i)} \oplus \tilde{g}_2^{(i)}$, where $\tilde{g}_1^{(i)} \in \tilde{G}_1$ and $\tilde{g}_2^{(i)} \in \tilde{G}_2$, $i = 1,2$. Let $\overline{F}_i$ be a closed fundamental region for $\tilde{G}_i$, $i = 1,2$, and define $\overline{F} = \overline{F}_1 \oplus \overline{F}_2$. Then the following statements are equivalent.

(1). $\tilde{g}_j^{(1)} \overset{\overline{F}_j}{\geq} \tilde{g}_j^{(2)}$, $j = 1,2$.

(2). $\tilde{g}^{(1)} \overset{\overline{F}}{\geq} \tilde{g}^{(2)}$.

(3). $A\tilde{g}^{(1)}A' \overset{A\overline{F}}{\geq} A\tilde{g}^{(2)}A'$.

Proof. That (2) and (3) are equivalent is a consequence of Lemma 3.2. An argument analogous to the one used in the proof of Lemma 3.4 establishes the equivalence of (1) and (2).||

Lemma 3.6. Let G, $\tilde{G}$, $\tilde{G}_1$, $\tilde{G}_2$, and A be as in Lemma 3.3. Let h be G-monotone increasing (decreasing) on V. Then $\tilde{h}(x) = h(Ax)$ is $\tilde{G}_1$-monotone increasing (decreasing) on $A'V$.

Proof. Suppose $x,y \in A'V$. Define $x = x_1 = x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in [A_1 \vdots 0]'V$ and $x_2, y_2 \in [0 \vdots A_2]'V$. Suppose $x_1 \overset{\tilde{G}_1}{\geq} y_1$ and $x_2 = y_2$. Then $x \overset{\tilde{G}}{\geq} y$ and consequently $Ax \overset{G}{\geq} Ay$. Thus $\tilde{h}(x) = h(Ax) \geq h(Ay) = \tilde{h}(y)$.||

Lemma 3.7. Let G, $\tilde{G}$, $\tilde{G}_1$, $\tilde{G}_2$, and A be as in Lemma 3.3. Let f be G-ordered with respect to $\overline{F}$ on G. Then $\tilde{f}(g) = f(AgA')$ is $\tilde{G}_1$-ordered with respect to $[A_1 \vdots 0]'\overline{F}$ on $\tilde{G}$.

Proof. Suppose $g_1, g_2 \in \tilde{G}$. Define $g_1 = g_1^{(1)} \oplus g_1^{(2)}$ and $g_2 = g_2^{(1)} \oplus g_2^{(2)}$, where $g_1^{(i)}, g_2^{(i)} \in \tilde{G}_i$, $i = 1,2$. Suppose that $g_1^{(1)} \overset{[A_1 \vdots 0]'\overline{F}}{\geq} g_1^{(2)}$ and that $g_2^{(1)} = g_2^{(2)}$ is the identity transformation on $[0 \vdots A_2]'V$. Then $g_1 \overset{A'\overline{F}}{\geq} g_2$ and consequently $Ag_1A' \overset{\overline{F}}{\geq} Ag_2A'$. Thus $\tilde{f}(g_1) = f(Ag_1A') \geq f(Ag_2A') = \tilde{f}(g_2)$.||

Results similar to the one shown in Lemma 3.7 exist for G-ordered functions on V^2 and also for functions G-ordered with respect to $\overline{F}$ on V.

We now present two theorems which embody our main applications in the area of hypothesis testing. Sufficient conditions are determined under which power functions of certain tests of multivariate hypotheses are G-monotone increasing or G-ordered with respect to a closed fundamental region $\overline{F}$.

Theorem 3.8. Let G, $\tilde{G}$, $\tilde{G}_1$, $\tilde{G}_2$, and A be as in Lemma 3.3. Let $K(\lambda,x)$, defined on $\Lambda \times X$, be an element of $K_{G,\mu}$ and suppose that f is G-monotone increasing on X.

Then the power function of a test of $H_o : A_1'\lambda = 0$ versus $H_1 : A_1'\lambda \neq 0$ based on $C_f = \{x \in X : f(x) > c_\alpha\}$ is $\tilde{G}_1$-monotone increasing on $A'\Lambda$.

Proof. Define $h(\lambda) = \int K(\lambda,x) I_{C_f}(x)\, d\mu(x)$, so that h is G-monotone increasing on Λ as a consequence of Theorem 4.15 or one of its corollaries. Then the power function $\tilde{h}(\lambda) = h(A\lambda)$ is $\tilde{G}_1$-monotone increasing on $A'\Lambda$ by Lemma 3.6.||

Theorem 3.9. Let G, $\tilde{G}$, $\tilde{G}_1$, $\tilde{G}_2$, and A be as in Lemma 3.3. Let K be G-ordered on $\Lambda \times X$ and be absolutely continuous with respect to a G-invariant measure μ. Let f be G-ordered with respect to $\overline{F}$ on X. Then the power function of a test of $H_o : A_1'\lambda \in \overline{F}$ versus $H_1 : A_1'\lambda \notin \overline{F}$ based on $C_f = \{x \in X : f(x) > c_\alpha\}$ is $\tilde{G}_1$-ordered with respect to $[A_1 \vdots 0]'\overline{F}$ on $A'\Lambda$.

Proof. Define $h(\lambda) = \int K(\lambda,x) I_{C_f}(x)\, d\mu(x)$, so that h is G-ordered with respect to $\overline{F}$ on Λ as a consequence of Corollary 3.14. Then the power function $\tilde{h}(\lambda) = h(A\lambda)$ is $\tilde{G}_1$-ordered with respect to $[A_1 \vdots 0]'\overline{F}$ on $A'\Lambda$ by Lemma 3.7 and the comment following it.||

Example 3.10. Suppose a random vector X has an elliptically-contoured density, i.e. the density of X, $K(\lambda,x)$, defined on $\Lambda \times X$, has the form $c \cdot g[(x - \lambda)'B(x - \lambda)]$, where B is positive definite and g is a decreasing function. Let $P = [r_1^{(1)} \ldots r_{n_1}^{(1)} \vdots \ldots \vdots r_1^{(k)} \ldots r_{n_k}^{(k)}] = [P_1 \vdots \ldots \vdots P_k]$ be a diagonalizer of B such that $\{r_1^{(i)},\ldots,r_{n_i}^{(i)}\}$ is a set of orthogonal eigenvectors corresponding to the eigenvalue α_i and basis for the subspace V_i, $i = 1,2,\ldots,k$. The density

$K(\lambda,x)$ is G-ordered for the group $G = O(V_1) \times O(V_2) \times \ldots \times O(V_k)$. Define

$\tilde{P}_j^{(1)} = [P_1 \vdots \ldots \vdots P_j]$ and $\tilde{P}_j^{(2)} = [P_{j+1} \vdots \ldots \vdots P_k]$, $i \le j \le k$, so that

$P'\Delta_G^* = [\tilde{P}_j^{(1)} \vdots 0]'\Delta_G^* \cup [0 \vdots \tilde{P}_j^{(2)}]'\Delta_G^*$. Note that the group generated by $P'\Delta_G^*$ is

$\tilde{G} = O(P'V_1) \times O(P'V_2) \times \ldots \times O(P'V_k)$, the group generated by $[\tilde{P}_j^{(1)} \vdots 0]'\Delta_G^*$ is

$\tilde{G}_1 = O(P'V_1) \times O(P'V_2) \times \ldots \times O(P'V_j)$, and the group generated by $[0 \vdots \tilde{P}_j^{(2)}]'\Delta_G^*$

is $\tilde{G}_2 = O(P'V_{j+1}) \times O(P'V_{j+2}) \times \ldots \times O(P'V_k)$. As a consequence of Theorem 3.8, the power function of any test of $H_o : \tilde{P}_j^{(1)'}\lambda = 0$ versus $H_1 : \tilde{P}_j^{(1)'}\lambda \ne 0$ based on $C_f = \{x \in X : f(x) > c_\alpha\}$, where f is G-monotone increasing, is $\tilde{G}_1$-monotone increasing. Tests of this form include tests of certain specified orthogonal contrasts.

As a specific case of Example 3.10, suppose the parameter space Λ is generated by the rows of $X_{3\times4}$, the design matrix for a simple one-way analysis of variance layout. Define

$$B = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}, \quad -\tfrac{1}{2} < \rho < 1.$$

Then

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, \quad \tilde{P}_2^{(1)} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/\sqrt{6} \end{bmatrix}, \text{ and } \tilde{P}_2^{(1)} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

Define $\lambda = X_{3\times4}\beta$, where $\beta \in R^4$. Then

$$\tilde{P}_2^{(1)'}\lambda = \begin{bmatrix} (\beta_2 - \beta_3)/\sqrt{2} \\ (\beta_2 + \beta_3 - 2\beta_4)/\sqrt{6} \end{bmatrix},$$

and any test of $H_o : \tilde{P}_2^{(1)'}\lambda = 0$ versus $H_1 : \tilde{P}_2^{(1)'} \neq 0$ based on $C_f = \{x \in \mathcal{X} : f(x) > c_\alpha\}$, where f is G-monotone increasing, has a $\tilde{G}_1$-monotone increasing power function. It this case $\tilde{G}_1$ is $O(R^2)$ and the test is actually equivalent to a test of $\beta_2 = \beta_3 = \beta_4$.

<u>Example 3.11</u>. For $\mu, \alpha_1, \alpha_2, \beta_1, \beta_2 \in R^1$, define

$$\lambda = \begin{bmatrix} \mu + \alpha_1 + \beta_1 \\ \mu + \alpha_1 + \beta_2 \\ \mu + \alpha_2 + \beta_1 \\ \mu + \alpha_2 + \beta_2 \end{bmatrix}.$$

Note that parameters of this form may arise from a two-way analysis of variance layout with no interaction. Suppose we desire to test $H_o : \alpha_1 = \alpha_2$ versus $H_1 : \alpha_1 \neq \alpha_2$. Define the orthogonal matrix

$$A = [A_1 \;\; A_2] = \left[\begin{array}{r|rrr} 1/2 & 1/\sqrt{2} & 0 & 1/2 \\ 1/2 & -1/\sqrt{2} & 0 & 1/2 \\ -1/2 & 0 & 1/\sqrt{2} & 1/2 \\ -1/2 & 0 & -1/\sqrt{2} & 1/2 \end{array}\right],$$

so that $A_1'\lambda = \alpha_1 - \alpha_2$. Let $\tilde{G}_1$ be the group of sign changes acting on $V_1 = \{x \in R^4 : x_1 \in R^1$ and $x_2 = x_3 = x_4 = 0\}$ and let $\tilde{G}_2$ be any reflection group acting on $V_2 = \{x \in R^4 : x_1 = 0$ and $x_2, x_3, x_4 \in R^1\}$. Define $\Delta^*_{\tilde{G}} = \Delta^*_{\tilde{G}_1} \cup \Delta^*_{\tilde{G}_2}$ and $\Delta^*_G = A\Delta^*_{\tilde{G}}$. If the observation vector X has a density $K(\lambda, x)$ belonging to $K_{G,\mu}$, then any test of $H_o : \alpha_1 = \alpha_2$ based on $C_f = \{x \in R^4 : f(x) > c_\alpha\}$, where f is G-monotone increasing, has a $\tilde{G}_1$-monotone increasing power function.

There are numerous examples in which Theorems 3.8 and 3.9 can be used to show monotonicity properties for power functions of tests of multivariate hypotheses. We may, for instance, apply both theorems to hypothesis tests involving the general linear model of the form $Y - X\beta + \varepsilon$, where ε is a random vector having density $g(x)$ on V. For the general linear model the parameter space Λ of Theorems 3.8 and 3.9 is the linear space spanned by the columns of X' and the density of the observations $K(\lambda x) = g(x - \lambda)$, where $\lambda = X\beta$.

Illustrations of the usefulness of Theorems 3.8 and 3.9 include a wealth of nonparametric tests. We may consider, for example, variables used in testing certain multivariate hypotheses to be the signs, ranks, or signed ranks of a set of observations. We then use Theorems 3.8 and 3.9 by allowing $K(\lambda,x)$ to be the frequency function of the signs, the ranks, or the signed ranks of a set of observations. Many well-known nonparametric tests of hypotheses can be formulated to fit the assumptions of Theorems 3.8 and 3.9, so that monotonicity properties for the power functions immediately ensue. These include the sign test of Fisher, the rank sum and the signed rank tests of Wilcoxon, and the test for equal treatment effects of Kruskal and Wallis. In each case the usual assumptions are such that the assumptions of Theorem 3.2.8 are satisfied. The ranklike tests of Ansari and Bradley and of Moses for determining equality of dispersion can be shown under the usual assumptions to have monotonic power functions using Theorem 3.8. The test for ordered alternatives against a null hypothesis of equal treatment effects presented by Jonckheere can be reformulated to fit the assumptions of Theorem 3.9. Consequently, the usual assumptions of the model dictate that Jonckheere's test has a G-ordered power function. It is important to note that in all these examples observations were assumed to be mutually independent. We have indicated, through Theorems 3.8 and 3.9, sufficient conditions on the joint distribution of the observations under which power functions are monotone. Independence of observations in most cases is not at all a necessary assumption.

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